# Symplectic Energy and Lagrangian Intersection Under Legendrian Deformations

**Hai-Long Her**

**Let $M$ be a compact symplectic manifold, and $L \subset M$ be a closed Lagrangian submanifold which can be lifted to a Legendrian submanifold in the contactization of $M$. For any Legendrian deformation of $L$ satisfying some given conditions, we get a new Lagrangian submanifold $L'$. We prove that the number of intersection $L \cap L'$ can be estimated from below by the sum of $\mathbb{Z}_2$-Betti numbers of $L$, provided they intersect transversally.**

## 1 Introduction.

In 1965, V. I. Arnold[A1][A2] formulated his famous conjectures concerning about the number of fixed points of Hamiltonian diffeomorphisms of any compact symplectic manifold and the number of intersection points of any Lagrangian submanifold with its Hamiltonian deformations in a symplectic manifold. More precisely, his conjectures can be written in topological terms as

$$\#\mathrm{Fix}(\psi_M) \geq \begin{cases} \text{sum of Betti numbers of } M, & \text{all fixed points are nondegenerate;} \\ \text{cuplength of } M, & \text{some fixed points maybe degenerate,} \end{cases}$$

and

$$\#(L \cap \psi_M(L)) \geq \begin{cases} \text{sum of Betti numbers of } L, & \text{intersection points are transverse;} \\ \text{cuplength of } L, & \text{maybe non} - \text{transverse,} \end{cases}$$

where $M$ is a symplectic manifold, $L \subset M$ is a Lagrangian submanifold, $\psi_M$ is a Hamiltonian diffeomorphism.

To prove these two conjectures, many works have been done, the pioneers of them are due to Conley-Zehnder[CZ], Gromov[G] and Floer[F1]-[F4]. Especially, Floer originally developed the seminal method, motivated by the variational method used by Conley and Zehnder and the elliptic PDE techniques introduced by Gromov, which is now called Floer homology theory, and solved many special cases of Arnold's conjectures. In 1996, Fukaya-Ono[FO] , Liu-Tian[LT] and Ruan[R] independently proved the first conjecture for general compact symplectic manifolds in the non-degenerate case. While the conjecture for general symplectic manifolds in the degenerate case is still open.

For the second conjecture, Floer[F1][F4] gave the proof under an additional assumption $\pi_2(M, L) = 0$. We write his result for the case that all intersections are transverse.

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**Floer's Theorem**. *Let $L$ be a closed Lagrangian submanifold of a compact(or tame) symplectic manifold $(M, \omega)$ satisfying $\pi_2(M, L) = 0$, and $\psi_M$ be a Hamiltonian diffeomorphism, then $\#(L \cap \psi_M(L)) \geq \dim H_*(L, \mathbb{Z}_2)$, if all intersections are transverse.*

In general, the condition $\pi_2(M, L) = 0$ can not be removed. For instance, let $L$ be a circle in $\mathbb{R}^2$, then $\pi_2(\mathbb{R}^2, L) \neq 0$, however, there always exists a Hamiltonian diffeomorphism which can translate $L$ arbitrarily far from its original position.

To prove his theorem, Floer introduced the so-called Floer homology group for Lagrangian pairs and showed that it is isomorphic to the homology of $L$ under the condition above. The definition of Floer homology for Lagrangian pairs was generalized by Oh[Oh2] in the class of monotone Lagragian submanifolds with minimal Maslov number being at least 3. However, for general Lagrangian pairs, the Floer homology is hard to define due to the bubbling off phenomenon and some essentially topological obstructions [FO$^3$], which is much different from the Hamiltonian fixed point case.

Therefore, if we want to throw away the additional assumption, we have to restrict the class of Hamiltonian diffeomorphisms. For the simplest case that $\psi_M$ is $C^0$-small perturbation of the identity, the Lagrangian intersection problem is equivalent to the one for zero sections of cotangent bundles, which is proved by Hofer[H1] and Laudenbach-Sikorav[LS]. Yu.V. Chekanov[C1][C2] also gave a version of Lagrangian intersection theorem which used the notion of symplectic energy introduced by Hofer[H2] (for $(\mathbb{R}^{2n}, \omega_0)$) and Lalonde-McDuff[LM] (for general symplectic manifolds). Following their notations, we denote by $\mathcal{H}(M)$ the space of compactly supported smooth functions on $[0,1] \times M$. Any $H \in \mathcal{H}(M)$ defines a time dependent Hamiltonian flow $\phi_H^t$ on $M$, all such time-1 maps $\{\phi_H^1,\ H \in \mathcal{H}(M)\}$ form a group, denoted by $Ham(M)$. Now we define a norm on $\mathcal{H}(M)$:

$$\|H\| = \int_0^1 (\max_{x \in M} H(t,x) - \min_{x \in M} H(t,x))dt,$$

and we can define the energy of a $\psi \in Ham(M)$ by

$$E(\psi) = \inf_H \{\|H\| \mid \psi = \phi_H^1,\ H \in \mathcal{H}(M)\}.$$

For a compact symplectic manifold $(M, \omega)$, there always exists an almost complex structure $J$ compatible with $\omega$, so $(M, \omega, J)$ is a compact almost complex manifold, we denote by $\mathcal{J}$ the set of all such $J$. Let $\sigma_S(M, J)$ and $\sigma_D(M, L, J)$ denote the minimal area of a $J$-holomorphic sphere in $M$ and of a $J$-holomorphic disc in $M$ with boundary in $L$, respectively. If there is no such $J$-holomorphic curve, these numbers will be infinity. Otherwise, minimums are obtained by the Gromov compactness theorem[G], and they are always positive. We write $\sigma(M, L, J) = \min(\sigma_S(M, J), \sigma_D(M, L, J))$, and $\sigma(M, L) = \sup_{J \in \mathcal{J}} \sigma(M, L, J)$. Then Chekanov showed the following theorem.

**Chekanov's Theorem**[C2]. *If $E(\psi) < \sigma(M, L)$, then $\#(L \cap \psi(L)) \geq \dim H_*(L, \mathbb{Z}_2)$, provided all intersections are transverse.*

*Remark.* For the non-transverse case, under similar assumptions, C.-G. Liu[L] also got an estimate for Lagrangian intersections by cup-length of $L$.

In this paper, we give an analogous Lagrangian intersection theorem, but the Hamiltonian deformation $\psi$ will be replaced by a "Legendrian deformation" $\tilde{\psi}$ (which will be explained in the sequel). In fact, K. Ono has shown such a result still under the assumption $\pi_2(M, L) = 0$.

Suppose that the symplectic structure $\omega$ is in an integral cohomology class, and there exists a principal circle bundle $\pi : N \to M$ with a connection so that the curvature form is $\omega$, that means for a connection form $\alpha$, one has $d\alpha = \pi^*\omega$. We see that the horizontal distribution $\xi = Ker(\alpha)$ is a co-oriented contact structure on $N$. We say $L$ satisfies the Bohr-Sommerfeld condition if $\alpha|_L$ is flat, or in other words, it can be lifted to a Legendrian submanifold $\Lambda$ in $N$. The following is Ono's result.

**Ono's Theorem**[On]. *Given a contact isotopy $\{\tilde{\psi}_t \mid 0 \le t \le 1\}$ on $N$, if $L$ is a Lagrangian submanifold of $M$ which can be lifted to a Legendrian submanifold $\Lambda$ in $N$, and $\pi_2(M, L) = 0$, then $\#(L \cap \pi \circ \tilde{\psi}_1(\Lambda)) \ge \dim H_*(L, \mathbb{Z}_2)$, provided $L$ and $\pi \circ \tilde{\psi}_1(\Lambda)$ intersect transversally.*

*Remark.* Since a Hamiltonian isotopy of $M$ can be lifted to a contact isotopy of $N$, Ono's theorem is a generalization of the previous Floer's theorem.

Eliashberg, Hofer, and Salamon[EHS] also independently obtained a result similar to Ono's theorem, they successfully overcome some difficulties due to the non-compactness of the symplectization manifold, while their arguments involve some complicated conditions for avoiding bubbling off.

In the present paper, we will throw away the assumption $\pi_2(M, L) = 0$ in Ono's theorem, at the same time, we will add a certain restrictive condition on the class of Legendrian deformation $\tilde{\psi}$. Firstly, we denote by $\tilde{L}$ the image of $\Lambda$ under the principal $S^1$-action on $N$. We denote by $(SN, \omega_\xi)$ the symplectization of the contact manifold $(N, \xi)$ with co-oriented contact strcture $\xi$, where the symplectic structure $\omega_\xi$ is induced from the standard 1-form of cotangent bundle $T^*N$. Then $\tilde{L}$ is a compact Lagrangian submanifold in $SN$. There is a natural projection $p : SN \to N$, and each section corresponds to a splitting $SN = N \times \mathbb{R}_+ = N \times (e^{-\infty}, +\infty]$. The contactomorphism $\tilde{\psi}$ can be lifted to a $\mathbb{R}_+$-equivariant Hamiltonian symplectomorphism $\Psi$ on $SN$. We denote $\mathcal{L} = p^{-1}(\Lambda)$, which is also a Lagrangian submanifold in $SN$. Then we can see that there is a 1-1 correspondence between $\tilde{L} \cap \Psi(\mathcal{L})$ and $L \cap \pi \circ \tilde{\psi}_1(\Lambda)$. However, the symplectization $SN$ is not compact. So the ordinary method of Floer Lagrangian intersection will be modified.

Following the argument of Ono[On], we can replace the symplectization $(SN,\ \omega_\xi)$ manifold by another symplectic manifold $(Q,\ \Omega)$, which may be considered as a symplectic filling in the negative end, so $Q$ coincides with $SN$ in the part $N \times [e^{-C}, +\infty] \supset \tilde{L}$, where $C > 0$ is a sufficiently large number. We note that $Q$ is a 2-plane bundle over $M$ and is diffeomorphic to the associated complex line bundle $N \times_{S^1} \mathbb{C}$. We define the compatible almost complex structure by $J'$ on $Q$ in the following way. Since $Q$ is the associated complex line bundle, the connection $\alpha$ on $N$ gives the decomposition of $TQ = \mathrm{Ver}(Q) \oplus \mathrm{Hor}(Q)$. And we have a $\omega$-compatible almost complex structure $J$ on $M$, then we lift $J$ to an almost complex structure on $\mathrm{Hor}(Q)$. Also we define the almost complex structure on each fiber by choosing the standard complex structure $J_0$ on complex plane $\mathbb{C}$. Then we let $J' = J \oplus J_0$, so $J'$ is uniquely determined by the choice of $J$ on $M$ and a connection on $N$. Furthermore, Ono (c.f. section 6 in [On]) showed that if we choose a

generic $J$ on $M$ in the sense of the construction of Floer homology for $(M, L)$, then $J'$ is also a regular or generic almost complex structure on $Q$. If we write $\Pi : Q \to M$ for the natural projection, then it is a $(J', J)$-holomorphic map. Therefore, a map $u = \Pi \circ \tilde{u} : \Sigma \to M$ is $J$-holomorphic if and only if $\tilde{u} : \Sigma \to M$ is $J'$-holomorphic. And we can see that, for $r > 1$, the image of the positive end $N \times \{r\} \subset SN$ in $Q$ is $J'$-*convex*. So we can choose the $\Omega$-compatible almost complex structure so that it coincides with $J'$ outside of a compact set. For simplicity, we still denote by $J$ this almost complex structure on $Q$ if without the danger of confusion.

Moreover, Ono also proved that there is an a priori $C^0$-bound for connecting orbits in $Q$ (Especially, all $J$-holomorphic curves which we concern are contained in a compact subset $K \subset Q$, while $K$ depends on the choice of the contact isotopy $\{\psi_t\}$ ), and the bubbling off argument can go through as in the case of compact symplectic manifold. So the minimal area of $J$-holomorphic spheres and $J$-holomorphic discs bounding Lagrangian submanifolds $\tilde{L}$ and $\mathcal{L}$ can be achieved, we denote it by

$$\sigma(Q, \tilde{L}, \mathcal{L}, J) = \min(\sigma_S(Q, J)|_K, \sigma_D(Q, \tilde{L}, J)|_K, \sigma_D(Q, \mathcal{L}, J)|_K, \sigma_D(Q, \mathcal{L}, \tilde{L}, J)|_K)$$

and

$$\sigma(Q, \tilde{L}, \mathcal{L}) = \sup_{J_M \in \mathcal{J}} \sigma(Q, \tilde{L}, \mathcal{L}, J = J_M \oplus J_0).$$

We will show that we can find a compactly supported Hamiltonian diffeomorphism $\Psi' \in Ham(Q)$ such that for a compact set $K$, the two images of $\Psi$ and $\Psi'$ coincide. For detailed explanation, we refer to [On] or the section 2. Now we denote a contactomorphism by $\psi$, then our main result is the following

**Theorem 1** *Let $M$ be a compact symplectic manifold, and $N$ be the principal $S^1$-bundle $\pi : N \to M$ defined above. Given a contact isotopy $\psi_t \mid 0 \leq t \leq 1\}$ on $N$, suppose $L$ is a closed Lagrangian submanifold of $M$ which can be lifted to a Legendrian submanifold $\Lambda$ in $N$, and $E(\Psi') < \sigma(Q, \tilde{L}, \mathcal{L})$, then $\#(L \cap \pi \circ \psi_1(\Lambda)) \geq \dim H_*(L, \mathbb{Z}_2)$, provided $L$ and $\pi \circ \psi_1(\Lambda)$ intersect transversally.*

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## 2 Preliminaries.

We introduce some fundamental concepts and facts in symplectic and contact geometry.

Given a $2n+1$-dimensional manifold $N$, we say $N$ is a contact manifold if there exists a contact structure $\xi$, which is a completely non-integrable tangent hyperplane distribution. It is obvious that $\xi$ can locally be defined by a 1-form $\alpha$, *i.e.* $\xi = \{\alpha = 0\}$ or $\xi = \ker \alpha$, satisfying $\alpha \wedge (d\alpha)^n \neq 0$. If the contact structure is co-orientable, then $\alpha$ can be global

defined. We only consider the co-oriented contact structure in this paper. The contact manifold is denoted by $(N, \xi)$, $\alpha$ is called a contact form. A diffeomorphism $\psi$ of $N$ is called a contactomorphism if it preserves the co-oriented contact structure $\xi$. $\{\psi_t,\ 0 \leq t \leq 1\}$ is called a contact isotopy, if $\psi_0 = \mathrm{id}$ and every $\psi_t$ is a contactomorphism. And $X_t = \frac{d\psi_t}{dt}$ is the contact vector field on $N$.

For any symplectic manifold $(M, \omega)$, there exists an almost complex structure $J$ on $M$. We say the almost complex is *compatible* with the symplectic manifold, if $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$, and $\omega(\cdot, J\cdot) > 0$, which can give the Riemannian metric on $M$.

Let $N$ be an oriented codimension 1 submanifold in an almost complex manifold $(Q, J)$, and $\xi_x$ be the maximal $J$-invariant subspace of the tangent space $T_xN$, then $\xi_x$ has codimension 1. And $N$ is said to be *$J$-convex* if for any defining 1-form $\alpha$ for $\xi$, we have $d\alpha(v, Jv) > 0$ for all non-zero $v \in \xi_x$. This implies $\xi$ is a contact structure on $N$. It is a fact that if $N$ is $J$-convex then no $J$-holomorphic curve in $Q$ can touch (or tangent to) $N$ from inside (from negative side) (c.f. [G], [M2]).

**Symplectization.**

We denote by $SN = S_\xi(N)$ the $\mathbb{R}_+$-subbundle of the cotangent bundle $T^*N$ whose fiber at $q \in N$ are all non-zero linear forms in $T_q^*N$ which is compatible with the contact hyperplane $\xi_q \subset T_qN$. There is a canonical 1-form $pdq$ on $T^*N$, and let $\alpha_\xi = pdq|_{SN}$, then $\omega_\xi = d\alpha_\xi$ is a symplectic structure on $SN$. Thus, we call $(SN,\ \omega_\xi)$ the symplectization of the contact manifold $(N,\ \xi)$. We see that a contact form $\alpha :\ N \to SN$ is a section of this $\mathbb{R}_+$-bundle $p :\ SN \to N$, hence we have a splitting $SN = N \times \mathbb{R}_+$.

An $n$-dimensional submanifold $\Lambda \subset (N, \xi)$ is called Legendrian if it is tangent to the distribution $\xi$, that is to say, $\Lambda$ is Legendrian iff $\alpha|_\Lambda = 0$. The preimage $\mathcal{L} = p^{-1}(\Lambda)$ is an $\mathbb{R}_+$-invariant Lagrangian cone in $(SN, \omega_\xi)$. Conversely, any Lagrangian cone in the symplectization projects onto a Legendrain submanifold in $(N, \xi)$.

$SN$ carries a canonical conformal symplectic $\mathbb{R}_+$-action. Every contactomorphism $\varphi$ uniquely lifts to a $\mathbb{R}_+$-equivariant symplectomorphism $\tilde{\varphi}$ of $SN$, which is also a Hamiltonian diffeomorphism of $SN$. Conversely, each $\mathbb{R}_+$-equivariant symplectomorphism of $SN$ projects to a contactomorphism of $(N, \xi)$. A function $F$ on $SN$ is called a contact Hamiltonian if it is homogeneous of degree 1, *i.e.* $F(cx) = cF(x)$ for all $c \in \mathbb{R}_+,\ x \in SN$.

The Hamiltonian flow generated by a contact Hamiltonian function is $\mathbb{R}_+$-equivariant, it defines a contact isotopy on $(N, \xi)$, therefore, any contact isotopy $\{\varphi_t\}$ is generated in this sense by a uniquely defined time-dependant contact Hamiltonian $F_t :\ SN \to \mathbb{R}$. There is a 1-1 correspondence between a contact vector field $X_t$ and a function on $N$: $f_t = \alpha(X_t)$, which is also called a contact Hamiltonian function.

**Contactization.**

If a symplectic manifold $(M,\ \omega)$ is exact, *i.e.* $\omega = d\alpha$, then it can be contactized, The contactization $C(M, \omega)$ is the manifold $N = M \times S^1$ (or $M \times \mathbb{R}$) endowed with the contact form $dz - \alpha$. Here we denote by $z$ the projection to the second factor and still denote by $\alpha$ its pull-back under the projection $N \to M$.

However, the contactization can be defined sometimes even when $\omega$ is not exact. Suppose that the form $\omega$ represents an integral cohomology class $[\omega] \in H^2(M)$. The contactization $C(M, \omega)$ of $(M, \omega)$ can be constructed as follows. Let $\pi :\ N \to M$ be a principal $S^1$-bundle with the Euler class equal to $[\omega]$. This bundle admits a connection whose cur-

vature form just is $\omega$. This connection can be viewed as a $S^1$-invariant 1-form $\alpha$ on $N$. The non-degeneracy of $\omega$ implies that $\alpha$ is a contact form and, therefore $\xi = \{\alpha = 0\}$ is a contact structure on $N$. The contact manifold $(N, \xi)$ is, by the definition, the contactization $C(M, \omega)$ of the symplectic manifold $(M, \omega)$. A change of the connection $\alpha$ leads to a contactomorphic manifold.

We note that a Hamiltonian vector field on $(M, \omega)$ can be lifted to a contact vector field on $N$. In fact, a Hamiltonian function $H$ on $M$ and its Hamiltonian vector field $X_H$ satisfy $dH = \iota(X_H)\omega$. And we know there exists a 1-1 correspondence between contact vector fields and functions on $N$, so we obtain a contact vector field $\tilde{X}_H$ on $N$ by $\alpha(\tilde{X}_H) = \pi^* H$. Also we have $\pi_* \tilde{X}_H = X_H$. Thus, any Hamiltonian isotopy on $M$ is lifted to a contact isotopy on $N$.

If $L \subset M$ is a Lagrangian submanifold, then the connection $\alpha$ over it is flat. The pull-back $\pi^{-1}(L) \subset N$ under the projection, which is also the image of the $S^1$-action of a Legendrian lift $\Lambda$, denoted by $\tilde{L}$, is a Lagrangian submanifold in $SN$ and is foliated by Legendrian leaves obtained by integrating the flat connection over $L$. If the holonomy defined by the connection $\alpha$ is integrable over $L$ then the Lagrangian submanifold $\tilde{L}$ is foliated by closed Legendrian submanifolds in $N$. In particularly, this is the case when the connection over $L$ is trivial. If this condition is satisfied then $L$ is called exact (Bohr-Sommerfeld condition). In this case the Lagrangian submanifold $\tilde{L}$ is foliated by closed Legendrian lifts of $L$.

A Legendrian submanifold $\Lambda \subset (N, \xi)$ has a neighborhood $U$ contactomorphic to the 1-jet space $J^1(\Lambda)$. Then $\tilde{L} \cap U$ can be identified under the contactomorphism with the so-called "0-wall": $W = \Lambda \times \mathbb{R} \subset J^1(\Lambda)$, which is just the set of 1-jets of function with 0 differential.

**Modify** $(SN, \omega_\xi)$**.**

Now, given a contact isotopy $\{\psi_t |\, 0 \leq t \leq 1\}$ of $(N, \xi)$. It can be lifted to a Hamiltonian isotopy $\{\Psi_t | 0 \leq t \leq 1\}$ of $SN$. Then, from the definition and properties listed above, we have a 1-1 correspondence between $L \cap \pi \circ \psi_1(\Lambda)$ and $\tilde{L} \cap \Psi_1(p^{-1}(\Lambda))$, also they coincide with $\tilde{L} \cap \psi_1(\Lambda)$, and all intersections are transversal. Therefore, it is natural to define Floer homology for such a pair of Lagrangian submanifolds $\tilde{L}$ and $\mathcal{L} = p^{-1}(\Lambda)$. However, as we all know, symplectization $SN$ is not compact, thus the ordinary method can not directly apply. Now, we adopt Ono's argument[On] to overcome this difficulty.

We see $N$ is compact, thus there exists large $C > 0$, such that the trace of $N$ under the isotopy $\{\Psi_t | 0 \leq t \leq 1\}$ is contained in a compact set $N \times [e^{-C}, e^C]$, and $N \times [e^{-C}, e^C]$ is disjoint from $\Psi_t(SN \setminus [e^{-D}, e^D])$, $t \in [0, 1]$, for some number $D > C$. So the part $N \times [e^{-D}, +\infty)$ is the domain we concern. The isotopy $\{\Psi_t\}$ is generated by a Hamiltonian $H : [0, 1] \times SN \to \mathbb{R}$. We can find another function $H'$, so that $H'$ equals $H$ on $N \times [e^{-C}, e^C]$, and equals zero outside of $N \times [e^{-D}, e^D]$. Then we get a new Hamiltonian isotopy $\{\Psi'_t | 0 \leq t \leq 1\}$ with compact support.

Since the boundary of the bundle $N \times [e^{-D-\epsilon}, +\infty)$ is of contact type, by symplectic filling techniques, the symplectization $(SN = N \times \mathbb{R}_+, \omega_\xi)$ can be replaced by a new symplectic manifold $(Q, \Omega)$, which is diffeomorphic to the associated complex line bundle $N \times_{S^1} \mathbb{C} \to M$. In fact, Ono showed there exists a symplectic embedding $\mathcal{F}$ from $N \times (e^{-D-\epsilon}, +\infty)$ into $(Q, \Omega)$(In fact, $\mathcal{F}$ is a symplectomorphism between $N \times (e^{-D-\epsilon}, +\infty)$ and $N \times_{S^1} \mathbb{C} - \{0 - \mathit{section}\}$, we refer to the appendix in [On] for details). Therefore,

we just study the Lagrangian intersection problem for $Q,\ \mathcal{F}\tilde{L},\ \mathcal{F}(\mathcal{L}\cap N\times(e^{-D-\epsilon},+\infty))$ under Hamiltonian isotopy $\Phi_t$ generated by a Hamiltonian defined on $Q$, which equals $H'\circ\mathcal{F}^{-1}$ on $N\times_{S^1}\mathbb{C}-\{0-section\}$, and equals zero on the 0-section. For simplicity, we still denote them by $\tilde{L},\ \mathcal{L},\ H$.

Also we notice that the positive end of $Q$ is $J$-convex, *i.e.* for a given $E>1$, $N\times\{E\}\subset Q$ is a $J$-convex codimension 1 submanifold. So there is no J-holomorphic curves can touch it, especially, there exists a $C^0$ bound for every J-holomorphic disc $u: D^2\to Q$ with boundary in Lagrangian submanifolds $\tilde{L}$ and $\Phi_t(\mathcal{L})$(also c.f. [On]). For general case, we consider $u:\Pi=\mathbb{R}\times[0,1]\to Q$ with $u(\tau,0)\subset\mathcal{L}$ and $u(\tau,1)\subset\tilde{L}$, $\tau\in\mathbb{R}$, which is regarded as the connecting orbit between $x_-(t)=lim_{\tau\to-\infty}u(\tau,t)$ and $x_+(t)=lim_{\tau\to+\infty}u(\tau,t)$, solving the perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial\tau}=-J\frac{\partial u}{\partial t}+\nabla H(t,u(\tau,t)).$$

In this situation, Gromov[G] showed how to define an almost complex structure $\tilde{J}_H$ on the product $\tilde{Q}=\Pi\times Q$, such that the $\tilde{J}_H$-holomorphic sections of $\tilde{Q}$ are precisely the graph $\tilde{u}$ of solutions of the equation above. We can see that $\tilde{Q}$ is $\tilde{J}_H$-convex, so there is a $C^0$-bound for $\tilde{J}_H$-holomorphic curves in $\tilde{Q}$, then the same thing happens to the connecting orbits in $Q$.

# 3 Variation and Functional.

From the discussion above, we know that we have got a symplectic manifold $(Q,\Omega)$, and two Lagrangian submanifolds $\tilde{L}$ and $\mathcal{L}$. Then we will establish a homology theory for the pair $(\tilde{L},\mathcal{L})$ in $Q$, and study critical points of the symplectic action functional defined on (some covering of) the space of paths in $Q$, starting from $\mathcal{L}$ with ends on $\tilde{L}$.

Let $H\in\mathcal{H}(Q)$ satisfy $\|H\|<\sigma(Q,\tilde{L},\mathcal{L},J)$, and $\Psi^t_{(s)},\ s\in[0,1]$, be the time-$t$ flow generated by Hamiltonian $sH$ (note that $\Psi^1_{(s)}$ is the lift of the contactomorphism $\psi^1_{(s)}$). And denote $\mathcal{L}_s=\Psi^1_{(s)}(\mathcal{L})$, $\Lambda_s=\psi^1_{(s)}(\Lambda)\subset N$. We suppose that $\tilde{L}$ intersects $\mathcal{L}_1$ transversally.

Let $\Sigma$ be the connected component of constant paths in the path space

$$\{\gamma\in C^\infty([0,1],Q)|\gamma(0)\in\mathcal{L},\ \gamma(1)\in\tilde{L}\}.$$

We define the closed 1-form $\alpha$ on $\Sigma$ by

$$\langle\alpha(\gamma),v\rangle=\int_0^1\Omega(\dot{\gamma}(t),v(t))dt,\ v(t)\in TQ|_{\gamma(t)},\ \forall\ t\in[0,1].$$

We also write the function $\theta:\ \Sigma\to\mathbb{R}$ as

$$\theta(\gamma)=-\int_0^1 H(t,\gamma(t))dt.$$

Note that the zeroes of $\alpha_s=\alpha+sd\theta$ are just time-1 trajectories generated by the flow $\Psi^t_{(s)}$ which start from $\mathcal{L}$ and end on $\tilde{L}$. If $\gamma$ is the zero of $\alpha_s$, then the ends of all $\gamma(1)$ are just the intersection points of $\tilde{L}$ with $\mathcal{L}_s$, which are 1-1 correspondent to the zeroes of $\alpha_s$. The purpose of this paper is to estimate from below the number of zeroes of $\alpha_1$ .

Since $H_t$ is compactly supported on $Q$, let $b_+ = \int_0^1 \max_{x\in Q} H(t,x)dt$, and $b_- = \int_0^1 \min_{x\in Q} H(t,x)dt$. Then $\|H\| = b_+ - b_-$, $-b_+ \leq \theta(\gamma) \leq -b_-$, for all $\gamma \in \Sigma$. We introduce the Riemannian structure on $\Sigma$ by the metric

$$(v_1, v_2) = \int_0^1 \Omega(v_1(t), Jv_2(t))dt.$$

Since

$$(grad_\alpha(\gamma), v) = \langle \alpha(\gamma), v\rangle = \int_0^1 \Omega(\dot\gamma(t), v(t))dt = \int_0^1 \Omega(J\dot\gamma(t), Jv(t))dt = (J\dot\gamma, v),$$

so the gradient of the closed 1-form $\alpha$ is given by $J\dot\gamma$, similarly, the gradient of the closed 1-form $\alpha_s$ is $grad_{\alpha_s} = J\dot\gamma - s\nabla H$.

Now, we consider the minimal covering $\pi : \tilde\Sigma \to \Sigma$ such that the form $\pi^*\alpha$ is exact, *i.e.* there is a functional $F$ on $\tilde\Sigma$, such that $\pi^*\alpha = dF$, and its structure group $\Gamma$ is free abelian. Denote $F_s = F + s(\theta\circ\pi)$, so $dF_s = \pi^*\alpha_s$. The gradient $\nabla F_s$ of the functional $F_s$, with respect to the lift of the Riemannian structure on $\Sigma$, is a $\Gamma$-invariant vector field on $\tilde\Sigma$, and $\pi_*\nabla F_s = grad_{\alpha_s}$. Then we consider the moduli space of thus gradient flows connecting a pair of critical points $(x_-, x_+)$ of $F_s$

$$M_s(x_-, x_+) = \{u : \mathbb{R} \to \tilde\Sigma| \ \frac{du(\tau)}{d\tau} = -\nabla F_s(u(\tau)),\ u \text{ is not constant}, \lim_{\tau\to\pm\infty} u(\tau) = x_\pm\}.$$

Denote by $\mathcal{M}_s = \bigcup_{x_\pm} M_s(x_-, x_+)$ the collection, and the nonparameterized space by $\hat M_s(x_-, x_+) = M_s(x_-, x_+)/\mathbb{R}$, and the natural quotient map $q : M_s \to \hat M_s$. Choosing a regular $\Omega$-compatible almost complex structure $J$ on $Q$ (c.f. [On])[1], we may assume that there is a dense set $T \subset [0,1]$ such that for all $s \in T$, $M_s(x_-, x_+)$ are finite dimensional smooth manifolds, consequently, $\tilde L$ intersects $\mathcal{L}_s$ transversally.

We define the length of a gradient trajectory $u \in M_s(x_-, x_+)$ by $l_s(u) = F_s(x_-) - F_s(x_+)$. If $\hat u \in \hat M_s$, then we define its length naturally by $l_s(\hat u) = l_s(u)$, where $\hat u = q\circ u$. Denote $\Pi = \mathbb{R}\times[0,1]$, then the map $\bar u : \Pi \to Q$, defined by $\bar u(\tau, t) = \pi(u(\tau))(t)$, satisfies the following perturbed Cauchy-Riemann equation

$$\frac{\partial \bar u(\tau,t)}{\partial\tau} = -J(\bar u(\tau,t))\frac{\partial \bar u(\tau,t)}{\partial t} + s\nabla H(t, \bar u(\tau,t)),$$

with limits

$$\lim_{\tau\to\pm\infty} \bar u(\tau,t) = \pi(x_\pm) = \bar x_\pm(t).$$

It is easy to see that $l_0(u) = \int_{-\infty}^{+\infty} u^*dF = \int_\Pi \bar u^*\Omega$.

If $u \in M_0$, then $\bar u$ is a J-holomorphic map from $\Pi$ to $Q$. From Oh's removing of boundary singularities theorem[Oh1], $\bar u$ can be extended to a J-holomorphic curve $\bar u' : (D^2, \partial^+ D^2, \partial^- D^2) \to (Q, \tilde L, \mathcal{L})$, where $D^2 = \bar\Pi$ is the two-point compactification of $\Pi$. Since $l_0(u) = \int_\Pi \bar u^*\Omega = \int_{D^2}(\bar u')^*\Omega$, we know that $l_0(u) \geq \sigma_D(Q, \tilde L, \mathcal{L}, J)$.

[1] Recall that the $J$ used here is just the $J' = J\oplus J_0$ given in the Introduction part, by generic choosing $\omega$-compatible almost complex structure $J$ on $M$ we can obtain the regular or generic $\Omega$-compatible structure $J'$ on $Q$. The arguments in [On] for $J'$-holomorphic maps can apply to our $H$-perturbed $J'$-holomorphic map by similar statements as those in [FHS]. We can overcome the similar problem which appears in the continuation argument of Section 6.

# 4 Define and Compute Homology for $C^0_\varepsilon$

We denote by $Y_s$ the set of critical points of $F_s$, and by $C_s$ the vector space spanned by $Y_s$ over $\mathbb{Z}_2$.

Since $Y_s$ is $\Gamma$-invariant, $C_s$ has a structure of free $K$-module with rank$= \#(\tilde{L} \cap \mathcal{L}_s)$, $s \in T$, where $K = \mathbb{Z}_2[\Gamma]$. Our aim in this section is to establish some homology for the complex $C_\varepsilon$, where $\varepsilon$ is small enough. We write the following definition similar as the one given by Chekanov[C2].

**Definition 4.1** *Fix $\delta > 0$, satisfying $\Delta := \|H\| + \delta < \sigma(Q, \tilde{L}, \mathcal{L}, J)$. A gradient trajectory $u \in M_s$ is said to be short if $l_s(u) \leq \Delta$, and be very short if $l_s(u) \leq \delta$.*

Now we denote the area by $A(u) = \int_\Pi \bar{u}^*\Omega$, and $h(u) = s\int_{-\infty}^{+\infty} u^* d(\theta \circ \pi)$, then still write $l(u) = l_s(u) = A(u) + h(u)$, we have

**Lemma 4.1** *If $u$ is very short i.e. $l(u) \leq \delta$, then the area $A(u) \leq \Delta$.*

Proof. since $\theta = -\int_0^1 H(t, \gamma(t))dt \in [-b_+, -b_-]$, then

$$h(u) = s\int_{-\infty}^{+\infty} u^* d(\theta \circ \pi) = s\theta(\pi(u(\tau)))|_{-\infty}^{+\infty} \geq s(b_- - b_+),$$

so

$$A(u) = l(u) - h(u) \leq \delta - (b_- - b_+) = \|H\| + \delta = \Delta.$$

Then we can prove the following key lemma[2]

**Lemma 4.2** . *For a small neighborhood $U$ of $\tilde{L}$ in $Q$, there exists a $\varepsilon_0 > 0$, such that for any positive $\varepsilon < \varepsilon_0$, every short gradient trajectory $u \in M_\varepsilon$ is very short, and for every short $u$ we have $\bar{u}(\Pi) \subset U$.*

Proof. We prove it by contradiction. For the first claim, we suppose there is a sequence $u_n \in M_{s_n}$ and a positive number $c$ with $\delta \leq c \leq \Delta$ so that when $s_n \to 0$ then $l_{s_n}(u_n) \to c$. By Gromov's compactness theorem, there are some subsequence of $\bar{u}_n = \pi(u_n)$ convergent to $\bar{u}_\infty$ which is a collection of $J$-holomorphic spheres and $J$-holomorphic discs bounding $\tilde{L}$ and/or $\mathcal{L}$. Then the total symplectic area of this limit collection is just $l_0(u_\infty) = c$ which by the assumption of Theorem 1 is lager than $\sigma(Q, \tilde{L}, \mathcal{L})$, but $c \leq \Delta < \sigma(Q, \tilde{L}, \mathcal{L})$, so the claim holds. For the second claim, the argument is similar. Note that if the image $\bar{u}_\infty(\Pi)$ of the limit collection is not contained in $U$, then at least one of the $J$-curve is not contained in $U$ which is nonconstant and its area will be larger than $\sigma(Q, \tilde{L}, \mathcal{L}, J) > \Delta$, this contradicts the (very) shortness condition. QED.

Then, we denote by $M'_\varepsilon$ ($\hat{M}'_\varepsilon$) $\subset M_\varepsilon$ ($\hat{M}_\varepsilon$) the set of all short gradient trajectories (nonparameterized short gradient trajectories). And we can define the $\mathbb{Z}_2$-linear map $\partial : C_\varepsilon \to C_\varepsilon$ by

$$\partial(x) = \sum_{y \in Y_\varepsilon} \#\{\text{isolated points of } \hat{M}'_\varepsilon(x, y)\}y,$$

[2] Actually, the lemma is essentially proved by Chekanov (c.f. Lemma 6 in [C2]), here we rewrite it in our settings with some modifications.

for $\forall\ x \in Y_\varepsilon$.

Let $\varepsilon \in T$ be sufficiently small and satisfy the conditions of lemma 4.2. Choose an element $x_0 \in Y_\varepsilon$, then we can define a subclass $Y_\varepsilon^0 \subset Y_\varepsilon$ by

$$Y_\varepsilon^0 = \{x \in Y_\varepsilon \mid |F_\varepsilon(x) - F_\varepsilon(x_0)| \le \delta\}.$$

Then we see that the projection $\pi$ bijectively maps $Y_\varepsilon^0$ onto the set of zeroes of the form $\alpha_\varepsilon$. And we get the bijection

$$Y_\varepsilon^0 \times \Gamma \to Y_\varepsilon : \quad (y, a) \mapsto a(y),$$

which induces the isomorphism $C_\varepsilon^0 \otimes K \to C_\varepsilon$, where $C_\varepsilon^0 \subset C_\varepsilon$ is spanned over $\mathbb{Z}_2$ by $Y_\varepsilon^0$. Now, for sufficiently small $\varepsilon \in T$, we can establish the homology for $(C_\varepsilon, \partial)$

**Lemma 4.3** $1^\circ$ *The map $\partial$ is $K$-linear, well defined, and $\partial(C_\varepsilon^0) \subset C_\varepsilon^0$;*
$2^\circ$ *If $\varepsilon \in T$ is sufficiently small, then $\partial^2 = 0$;*
$3^\circ$ *The homology $H(C_\varepsilon^0, \partial) \cong H_*(\Lambda, \mathbb{Z}_2)$.*

Proof. $1^\circ$ Since the gradient flow is $\Gamma$-invariant, $\partial$ is naturally $K$-linear. We know that the bubbling off can not occur. Indeed, since $\varepsilon$ is sufficiently small, then $u \in M_\varepsilon$ is very short, $l_\varepsilon(u) \le \delta$, by the lemma 4.1, the area $A(u) \le \Delta < \sigma(Q, \tilde{L}, \mathcal{L}, J)$, and from the assumption in our theorem, the area of any J-holomorphic sphere or J-holomorphic disc bounding $\tilde{L}$ and $\mathcal{L}$ is larger than $\sigma(Q, \tilde{L}, \mathcal{L}, J)$. Thus, $\hat{M}'_\varepsilon(x, y)$ is compact and the number of its isolated points is finite.

$2^\circ$ Suppose $\varepsilon \in T$ satisfy the conditions in lemma 4.2. If $\|H\| = 0$, $\Delta = \delta$, then $H \equiv const.$ and $\psi_H \equiv id$, it is a trivial case. If $\|H\| > 0$, we can always choose a fixed $\delta < \frac{1}{2}\|H\| < \frac{1}{2}\Delta$. Consider a pair of isolated trajectories $u_1 \in \hat{M}'_\varepsilon(x, y)$, $u_2 \in \hat{M}'_\varepsilon(y, z)$. Then there exists a unique 1-dimensional connected component $\mathcal{C} \subset \hat{M}_\varepsilon(x, z)$ such that $(u_1, u_2)$ is one of the two ends of compactification of $\mathcal{C}$(c.f. [F1]). Since the length is additive under gluing, we have, for $\forall u \in \mathcal{C}$, $l_\varepsilon(u) = l_\varepsilon(u_1) + l_\varepsilon(u_2) < 2\delta < \Delta$. By lemma 4.2, $u$ is also very short. From the lemma 4.1, we know the bubbling off doesn't occur, too. Then the other end of $\mathcal{C}$ can be compactified by a pair of isolated trajectories $u'_1 \in \hat{M}_\varepsilon(x, y)$, $u'_2 \in \hat{M}_\varepsilon(y, z)$. Also $l_\varepsilon(u'_1) + l_\varepsilon(u'_2) = l_\varepsilon(u_1) + l_\varepsilon(u_2) < \Delta$, thus $u'_1$, $u'_2$ are short trajectories. So we know that, for $x, z \in Y_\varepsilon$, the number of isolated trajectories $(u_1, u_2) \in \hat{M}'_\varepsilon(x, y) \times \hat{M}'_\varepsilon(y, z)$ is even, *i.e.* $\partial^2(x) = 0$.

$3^\circ$ From the lemma 4.2, we know that for any $u \in \hat{M}'_\varepsilon(x, y)$, $\bar{u}(\bar{\Pi}) \subset U$. That is to say, if $\varepsilon$ is small enough, then $\Lambda_\varepsilon = \psi^1_{(\varepsilon)}(\Lambda)$ is always contained in a small neighborhood $U' = U \cap N$ of the Legendrian submanifold $\Lambda$ in $N$. By Darboux's theorem, $U'$ is contactomorphic to a neighborhood of the 0-section in the 1-jet space $J^1(\Lambda)$. This contactomorphism moves $\tilde{L} \cap U'$ onto the"0-wall" $W$, *i.e.* the space of 1-jets of functions with 0 differential. Thus a Legendrian submanifold $\Lambda'$, which is $C^1$-close to $\Lambda$ and transverse to $W$, corresponds to a Morse function $\beta : \Lambda \to \mathbb{R}$ so that the intersection points of $\tilde{L}$ and $\Lambda'$ are in 1-1 correspondence with the critical points of the function $\beta$. We can explicitly choose a metric on $\Lambda$ and a generic almost complex structure $J$ (recall the footnote in section 3) on the symplectization $SN$ in such a way that the gradient trajectories in $\mathcal{M}_\varepsilon$ would be in 1-1 correspondence with the gradient trajectories of the function $\beta$ connecting the

corresponding critical points of this function. Thus we can identify our complex $C_\varepsilon^0$ with the Morse chain complex for the function $\beta$ (here we may also reduce the problem to Lagrangian intersections in $M$ by applying continuation argument and projecting the manifold to $M$, the method of equating Floer and Morse complex is standard, we refer the reader to [S]), so we have an isomorphism $H(C_\varepsilon^0, \partial) \cong H_*(\Lambda, \mathbb{Z}_2)$.

# 5 Homology Algebra.

Under the condition $\pi_2(Q, \cdot) = 0$ or the monotonicity assumption[F1][Oh2], the Floer homology of the complex $C_s,\ s \in T \subset [0,1]$, can be defined, *i.e.* $HF_*(C_s, \partial)$. Then we can use the classical continuation method (c.f. [F4][M1][Oh2]) to prove the isomorphism between $HF_*(C_\varepsilon, \partial)$ and $HF_*(C_1, \partial)$, that means to construct chain homotopy $\Phi'\Phi \sim id_\varepsilon$ (and $\Phi\Phi' \sim id_1$), where $\Phi: \ (C_\varepsilon, \partial) \to (C_1, \partial)$, $\Phi': (C_1, \partial) \to (C_\varepsilon, \partial)$ are chain homomorphisms defined similarly as the definition of $\partial$, except for considering the moduli space of continuation trajectories. That is to say, in order to prove $\Phi'\Phi \sim id_\varepsilon$, we should show there exists a chain homomorphism $\mathbf{h}: \ (C_\varepsilon, \partial) \to (C_\varepsilon, \partial)$, so that

$$\Phi'\Phi - id = \mathbf{h}\partial - \partial\mathbf{h}.$$

However, in general case, we can not define appropriately any homology for $C_s$ unless $s$ is small enough. Then we may only prove a weaker "homotopy", which is called $\lambda$-*homotopy* by Chekanov. In fact, for the aim of estimating from below the number of critical points of the functional $F_s$, this $\lambda$-homotopy is enough.

We shall use the following homology algebraic result introduced and proved by Chekanov [C2].

Let $\Gamma$ be a free abelian group equipped with a monomorphism $\lambda: \ \Gamma \to \mathbb{R}$, which we call a *weight function.* Denote

$$\Gamma^+ = \{a \in \Gamma | \lambda(a) > 0\} \qquad \Gamma^- = \{a \in \Gamma | \lambda(a) < 0\}$$

Let $k$ be a communicative ring. Consider the group ring $K = k[\Gamma]$. For a $k$-module $M$, we have the natural decomposition $M \otimes K = M^+ \oplus M^0 \oplus M^-$. where $M^+ = \Gamma^+(M)$, $M^0 = M$, $M^- = \Gamma^-(M)$. Consider the projections

$$p^+: \ M \otimes K \to M^+ \oplus M^0, \qquad p^-: \ M \otimes K \to M^0 \oplus M^-.$$

Assume that $(M, \partial)$ is a differential $k$-module, then $\partial$ naturally extends to a $K$-linear differential on $M \otimes K$.

**Definition 5.1** *We say two linear maps $\phi_0,\ \phi_1: \ M \otimes K \to M \otimes K$ are $\lambda$-homotopic if there exists a $K$-linear map $\mathbf{h}: \ M \otimes K \to M \otimes K$ such that*

$$p^+(\phi_0 - \phi_1 + \mathbf{h}\partial + \partial\mathbf{h})p^- = 0.$$

**Lemma 5.1** [C2] *Let $\lambda$ be a weight function on a free abelian group $\Gamma$. Assume $(M, \partial)$ to be a differential $k$-module and $N$ to be a $K$-module, where $K = k[\Gamma]$. if the maps $\Phi^+: \ M \otimes K \to N$, $\Phi^-: \ N \to M \otimes K$ are $K$-linear and $\Phi^-\Phi^+$ is $\lambda$-homotopic to the identity, then* $\mathrm{rank}_K N \geq \mathrm{rank}_k H(M, \partial)$.

## 6 Proof of Theorem 1.

Given a $(s_-, s_+)$ continuation function $\rho : \mathbb{R} \to [0,1]$ satisfying

$$\rho(\tau) = \begin{cases} s_-, & \text{if } \tau < -r, \\ s_+, & \text{if } \tau > r, \end{cases}$$

where $r \in \mathbb{R}$, we can define the moduli space of continuation trajectories

$$M_\rho(x_-, x_+) = \{u : \mathbb{R} \to \tilde{\Sigma} \mid \frac{du(\tau)}{d\tau} = -\nabla F_{\rho(\tau)}(u(\tau)), \lim_{\tau \to \pm\infty} u(\tau) = x_\pm\},$$

where $x_\pm \in Y_{s_\pm}$. And we denote the collection by $\mathcal{M}_\rho = \bigcup_{x_-, x_+} M_\rho(x_-, x_+)$. The length of a continuation trajectory is defined by $l_\rho(u) = F_{s_-}(x_-) - F_{s_+}(x_+)$.

Choose a monotone $(\varepsilon, 1)$ continuation function $\rho_+$ and a monotone $(1, \varepsilon)$ continuation function $\rho_-$. For generic $H$, $M_{\rho_\pm}(x_-, x_+)$ are smooth manifolds. We will say a continuation trajectory $u \in M_{\rho_+}(x_-, x_+)$ (or $u \in M_{\rho_-}(x_-, x_+)$) is short if $l_{\rho_+}(u) \leq \delta + (1-\varepsilon)b_+$ (resp. $l_{\rho_-}(u) \leq \delta + (\varepsilon - 1)b_-$). The subspace of all short trajectories is denoted by $M'_{\rho_\pm}(x_-, x_+) \subset \mathcal{M}_{\rho_\pm}$.

Then, under an ideal assumption[3], *i.e.* if there is no a sequence of continuation trajectories reaches the negative end (*i.e* the zero section of $Q \to M$), we can simply construct the $\mathbb{Z}_2$-linear continuation map $\Phi^+ : C_\varepsilon \to C_1,\ \Phi^- : C_1 \to C_\varepsilon$ as

$$\Phi^+(x) = \sum_{y \in Y_1} \#\{\text{isolated points of } M'_{\rho_+}(x, y)\} y,$$

$$\Phi^-(y) = \sum_{z \in Y_\varepsilon} \#\{\text{isolated points of } M'_{\rho_-}(x, z)\} z,$$

where $x \in Y_\varepsilon$. The following lemma implies that the definition of $\Phi^\pm$ above is sound.

**Lemma 6.1** *If $u \in M_{\rho_+}(x_-, x_+)$, then $l_{\rho_+}(u) \geq (1-\varepsilon)b_-$. If $u \in M_{\rho_-}(x_-, x_+)$, then $l_{\rho_-}(u) \geq (\varepsilon - 1)b_+$. And the sum in the definition of $\Phi^\pm$ is finite.*

Proof. Recall $F_s = F + s\theta \circ \pi$, $s \in [0,1]$. For a $(s_-, s_+)$ continuation function $\rho : \mathbb{R} \to [0,1]$, we have $F_{\rho(\tau)} = F + \rho(\tau)\theta \circ \pi$. So the length of a continuation trajectory is

$$l_\rho(u) = F_{s_-}(x_-) - F_{s_+}(x_+) = -\int_{-\infty}^{+\infty} u^* dF_{\rho(\tau)} = -\int_{-\infty}^{+\infty} u^* dF - \int_{-\infty}^{+\infty} u^* d(\rho(\tau)\theta \circ \pi)$$
$$= A(u) + h(u),$$

where we denote

$$A(u) = -\int_{-\infty}^{+\infty} u^* dF = \int_\Pi \bar{u}^* \Omega = \int_{-\infty}^{+\infty} (\frac{du(\tau)}{d\tau}, \frac{du(\tau)}{d\tau}) d\tau = \int_{-\infty}^{+\infty} \|\nabla F_{\rho(\tau)}\|^2 d\tau \geq 0,$$

and

$$h(u) = -\int_{-\infty}^{+\infty} u^* d(\rho(\tau)\theta \circ \pi) = -\int_{-\infty}^{+\infty} \frac{d\varrho(\tau)}{d\tau} \theta(\pi(u(\tau))) d\tau.$$

[3]For general case, we have to modify the continuation map. The possibility that there exists a sequence of continuation trajectories reaching the negative end was pointed out to the author by one of referees.

Recall that

$$\theta = -\int_0^1 H dt \in [-b_+, -b_-],$$

if $u \in M_{\rho_+}(x_-, x_+)$,

$$l(u) \geq h(u) \geq (1-\varepsilon)b_-;$$

if $u \in M_{\rho_-}(x_-, x_+)$,

$$l(u) \geq h(u) \geq (\varepsilon - 1)b_+.$$

Thus, For a short trajectory $u \in M'_{\rho_\pm}(x_-, x_+)$,

$$A(u) = l(u) - h(u) \leq \delta + (1-\varepsilon)(b_+ - b_-) = \|H\| + \delta = \Delta.$$

Since $A(u) = \int_\Pi \bar{u}^*\Omega \leq \Delta < \sigma(Q, \tilde{L}, J)$, by Gromov's arguments, no bubbling can occur, then spaces $M'_{\rho_\pm}(x_-, x_+)$ are compact, and the finiteness of $\#\{\text{isolated points of } M'_{\rho_\pm}(x,y)\}$ is verified.

However, in general case, the ideal assumption is not always satisfied. If a sequence of continuation trajectories converges to a curve reaching the negative end of $\mathcal{L}$, the boundary of moduli space of continuation trajectories will contain such curve. So we need modify the definition of $\Phi^\pm$ to get a well-defined continuation map. In fact, Ono (c.f. [On] P.218) had dealt with a similar problem by considering the algebraic intersection number of the continuation trajectories with the zero section $O_M$ of $Q \to M$. Under Ono's assumption $\pi_2(M,L) = 0$, bubbling off of holomorphic discs contained in the zero section of $Q$ with boundary on $L$ never occurs. In our case, since we have the bound for energy, such kind of bubbling off of discs is also avoided. Thus, we can define homomorphisms $\Phi_k^+ : C_\varepsilon \to C_1,\ \Phi_k^- : C_1 \to C_\varepsilon$ as

$$\Phi_k^+(x) = \sum_{y \in Y_1} Int_{2,k}^+(x,y)y,$$

$$\Phi_k^-(y) = \sum_{z \in Y_\varepsilon} Int_{2,k}^-(y,z)z,$$

where $Int_{2,k}^+(x,y)$ ($Int_{2,k}^-(y,z)$) is the mod-2 number of the isolated points $u$ in the continuational moduli space $M'_{\rho_+}(x,y)$ (resp. $M'_{\rho_-}(y,z)$) which have the algebraic intersection number $u \cdot O_M = \frac{k}{2}$.

With the restriction of bound of energy, we can make similar discussions as in [On] to get the finiteness of $Int_{2,0}^\pm$. So $\Phi_0^\pm$ are just our favorite continuation maps. We will not list the detailed arguments here and refer the reader to [On] for original discussion. In the following, we will still denote the continuation maps by $\Phi^\pm$ for simplicity.

Then we can use the homology algebraic result listed in the section 5 to prove the theorem 1, provided there exists a $\lambda$-homotopy between $\Phi^-\Phi^+$ and the identity.

- *Prove the theorem 1.*

Now in our case, let $k = \mathbb{Z}_2,\ K = \mathbb{Z}_2[\Gamma],\ M = C_\varepsilon^0,\ M \otimes K = C_\varepsilon,\ N = C_1$, and $\Gamma$ be the structure group of the covering. The weight function $\lambda : \Gamma \to \mathbb{R}$ can be defined as $\lambda(a) = F(a(x)) - F(x)$. We also have decompositions $Y_\varepsilon = Y_\varepsilon^+ \cup Y_\varepsilon^0 \cup Y_\varepsilon^-$, $C_\varepsilon = C_\varepsilon^+ \oplus C_\varepsilon^0 \oplus C_\varepsilon^-$, where $Y_\varepsilon^\pm = \Gamma^\pm(Y_\varepsilon^0)$.

Assume that we have got a $\lambda$-homotopy $\mathbf{h}: C_\varepsilon \to C_\varepsilon$, then by lemma 5.1 and lemma 6.1 we have

$$\#(L \cap \pi \circ \psi_1(\Lambda)) = \#(\tilde{L} \cap \psi_1(\Lambda)) = \#(\tilde{L} \cap \Psi_1(\mathcal{L})) = \mathrm{rank}_K C_1$$

$$\geq \mathrm{rank}_k H(C_\varepsilon^0, \partial) = \mathrm{dim} H_*(\Lambda, \mathbb{Z}_2) = \mathrm{dim} H_*(L, \mathbb{Z}_2).$$

This finishes the proof of the Theorem 1.

In the rest of this section, we show a sketchy proof of the existence of $\lambda$-homotopy.

**Lemma 6.2** *There exists a $\lambda$-homotopy $\mathbf{h}: C_\varepsilon \to C_\varepsilon$ between $\Phi^-\Phi^+$ and the identity.*

Proof. We follow the arguments of Chekanov and state his main thought. For constructing the homomorphism $\mathbf{h}$, we use a family of $(\varepsilon, \varepsilon)$ continuation functions $\mu_c,\ c \in [0, +\infty)$ satisfying
1) $\mu_0(\tau) \equiv \varepsilon$,
2) $\frac{du_c(\tau)}{d\tau} \geq 0$, if $\tau < 0$; $\frac{du_c(\tau)}{d\tau} \leq 0$, if $\tau > 0$,
3) $c \mapsto \mu_c(0)$ is a monotone map from $[0, +\infty)$ onto $[\varepsilon, 1]$,
4) when $c$ is large enough $\mu_c(\tau) = \begin{cases} \rho_+(\tau + c), & \text{if } \tau \leq 0; \\ \rho_-(\tau - c), & \text{if } \tau \geq 0. \end{cases}$

Then we denote the moduli space

$$M_\mu(x_-, x_+) = \{(c, u) | u \in M_{\mu_c}(x_-, x_+)\},\ x_\pm \in Y_\varepsilon.$$

For generic $H$, $M_\mu(x_-, x_+)$ are smooth manifolds.

And like the arguments for $(\varepsilon, 1)$ or $(1, \varepsilon)$-continuation trajectories shown before in this section, under a similar ideal assumption, *i.e.* no any sequence of $\mu_c$-continuation trajectories reaches the negative end of $\mathcal{L}$, we can define the $\mathbb{Z}_2$-linear map for $C_\varepsilon^0$ as[4]

$$\mathbf{h}(x) = \sum_{y \in Y_\varepsilon^0} \#\{\text{isolated points of } M'_\mu(x, y)\} y, \quad x \in Y_\varepsilon^0$$

where $M'_\mu(x, y)$ is the subset of the moduli space $M_\mu(x, y)$ which contains only short $\mu_c$-continuation trajectories, a $\mu_c$-continuation trajectory $u \in M_{\mu_c}(x_-, x_+)$ is called short if its length $l(u) = l_{\mu_c}(u) \leq \delta$. Moreover, For any $u \in M_{\mu_c}(x_-, x_+)$, we have $l_{\mu_c}(u) = A(u) + h(u) \geq h(u) \geq (\mu_c(0) - \varepsilon)b_- + (\varepsilon - \mu_c(0))b_+ \geq b_- - b_+ = -\|H\|$, and if $l(u) \leq \delta$, then $A(u) = l(u) - h(u) \leq \delta + \|H\| \leq \Delta$.

The map $\mathbf{h}$ can be extended naturally to a $K$-linear map on $C_\varepsilon$. Since for $u \in M'_\mu(x, y)$, $l_{\mu_c}(u) \leq \delta,\ A(u) \leq \Delta$, the bubbling off does not occur, $M'_\mu(x, y)$ is compact and the sum is finite, thus the map $\mathbf{h}$ is well defined.

To prove $\mathbf{h}$ is a $\lambda$-homotopy, we have to verify

$$p^+(x + \Phi^-\Phi^+ x + \mathbf{h}\partial x + \partial \mathbf{h} x) = 0,$$

for $\forall\ x \in Y_\varepsilon^0 \cup Y_\varepsilon^-$. This will follow from the standard gluing argument involving the ends of the 1-dimensional part $\aleph$ of $M_\mu(x, z),\ z \in Y_\varepsilon^+ \cup Y_\varepsilon^0$. Since $x \in Y_\varepsilon^0 \cup Y_\varepsilon^-,\ z \in Y_\varepsilon^+ \cup Y_\varepsilon^0$,

[4] Otherwise, we will again adopt the Ono's argument to take into consideration of the algebraic intersection number, the way of modifying the definition of the map $\mathbf{h}$ is similar as the way of modifying $\Phi^\pm$ we have stated before. For simplicity, we just show the argument under this ideal assumption.

and $\varepsilon$ is small enough, we know $l(u) \leq \delta$ for $u \in \aleph$. Indeed, $l(u) = F_\varepsilon(x) - F_\varepsilon(z)$, and there exist $x'$ and $z'$ in $Y_\varepsilon^0$ and $a \in \Gamma^+ \cup \Gamma^0$, $b \in \Gamma^0 \cup \Gamma^-$ such that $z = a(z')$, $x = b(x')$, and $F_\varepsilon(x) - F_\varepsilon(x') = \lambda(b) \leq 0$, $F_\varepsilon(z') - F_\varepsilon(z) = -\lambda(a) \leq 0$, also we know that $F_\varepsilon(x') - F_\varepsilon(z') \leq \delta$ since $x', z' \in Y_\varepsilon^0$, this implies $l(u) \leq \delta$, so $A(u) \leq \Delta$[5]. This disappears the bubbling off. Then the compactification of $\aleph$ shows that the left hand side of the formula above has the expression

$$\sum_{z \in Y_\varepsilon^+ \cup Y_\varepsilon^0} \#\{S(x,z)\} z,$$

and the number $\#\{S(x,z)\}$ is even (the one more thing we should verify than the standard gluing argument is to prove the other ends of the compactification which are pairs of continuation trajectories are still all short, this is not difficult to do[6]). This ends the proof of the lemma and the theorem 1. For more details, the reader may refer to [C2][F4][M1].

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[5] From the proof of $l(u) \leq \delta$ here, the reader can see why we would only verify the so called $\lambda$-homotopy.

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Institute of Mathematical Science
Nanjing University
Nanjing 210093, P.R.China[7]
And
Chern Institute of Mathematics
Nankai University
Tianjin 300071, P.R.China
E-mail: hailongher@ims.nju.edu.cn

[7]This is the present postal address of the author.